

k -CLEAN MONOMIAL IDEALS

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ABSTRACT. In this paper, we introduce the concept of k -clean monomial ideals as an extension of clean monomial ideals and present some homological and combinatorial properties of them. Using the hierarchal structure of k -clean ideals, we show that a $(d-1)$ -dimensional simplicial complex is k -decomposable if and only if its Stanley-Reisner ideal is k -clean, where $k \leq d-1$. We prove that the classes of monomial ideals like monomial complete intersection ideals, Cohen-Macaulay monomial ideals of codimension 2 and symbolic powers of Stanley-Reisner ideals of matroid complexes are k -clean for all $k \geq 0$.

INTRODUCTION

Let R be a Noetherian ring and M be a finitely generated R -module. It is well known that there exists a so called prime filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_{r-1} \subset M_r = M$$

that is such that $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \text{Supp}(M)$. We call any such filtration of M a **prime filtration**. Set $\text{Supp}(\mathcal{F}) = \{P_1, \dots, P_r\}$. Let $\text{Min}(M)$ denote the set of minimal prime ideals in $\text{Supp}(M)$. If I is an ideal of R then we set $\text{min}(I) = \text{Min}(R/I)$. Dress [7] calls a prime filtration \mathcal{F} of M **clean** if $\text{Supp}(\mathcal{F}) = \text{Min}(M)$. The module M is called clean, if M admits a clean filtration and R is clean if it is a clean module over itself.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminate over a field K . Let Δ be a simplicial complex on the vertex set $[n] = \{1, 2, \dots, n\}$. Dress [7] showed that Δ is (non-pure) shellable in the sense of Björner and Wachs [3], if and only if the Stanley-Reisner ring S/I_Δ is clean. The result of Dress is, in fact, the algebraic counterpart of shellability for simplicial complexes. Some subclasses of shellable complexes are k -decomposable simplicial complexes which were introduced by Billera and Provan [2] on pure simplicial complexes and then by Woodroffe [31] on not necessarily pure ones. Simon in [25] introduced “completed clean ideal trees” as an algebraic counterpart of pure k -decomposable complexes. Actually, in the sense of Simon, the Stanley-Reisner ideal of a k -decomposable complex is completed clean ideal tree.

Let $I \subset S$ be a monomial ideal. We call I Cohen-Macaulay (clean) if the quotient ring S/I has this property. In this paper, we define the concept of k -clean monomial ideals. The class of k -clean monomial ideals are, actually, subclass of clean monomial ideals. It is the aim of this paper to study the properties of k -clean monomial ideals and describe relations between these ideals and k -decomposable

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simplicial complexes. Moreover, some classes of k -clean monomial ideals are introduced. Also, some results of [1, 16] are extended.

In Section 2, we introduce k -clean monomial ideals. We show that k -clean monomial ideals are clean and, also, every clean monomial ideal is k -clean for some $k \geq 0$ (see Theorem 2.4). In Section 3, we discuss some of basic properties of k -clean ideals. Some homological invariants of k -clean monomial ideals like depth and Castelnuovo-Mumford regularity are described in this section. In the fourth section, we show that a $(d-1)$ -dimensional simplicial complex Δ is k -decomposable if and only if its associated Stanley-Reisner ideal is k -clean, where $k \leq d$ (see Theorem 4.1). The last section is devoted to presenting some examples of k -clean monomial ideals. We show that irreducible monomial ideals and monomial complete intersection ideals are k -clean, for all $k \geq 0$ (see Theorems 5.1 and 5.2). Then by showing that Cohen-Macaulay monomial ideals of codimension 2 (see Theorem 5.4) are k -clean, we improve Proposition 1.4. of [16]. Finally, in Theorem 5.5, we show that symbolic powers of Stanley-Reisner ideals of matroid complexes are k -clean for all $k \geq 0$. In this way, we improve Theorem 2.1 of [1].

1. PRELIMINARIES

Let Δ be a simplicial complex of dimension $d-1$ with the vertex set $[n] := \{1, 2, \dots, n\}$. Let K be a field. The Stanley-Reisner monomial ideal of Δ is denoted by I_Δ and it is a squarefree monomial ideal in the polynomial ring $S = K[x_1, \dots, x_n]$ generated by the monomials $\mathbf{x}^F = \prod_{i \in F} x_i$ which F is a non-face in Δ . The quotient ring S/I_Δ is called the **face ring** or **Stanley-Reisner ring** of Δ . If $\mathcal{F}(\Delta) = \{F_1, \dots, F_r\}$ is the set of maximal faces (facets) of Δ then we set $\Delta = \langle F_1, \dots, F_r \rangle$.

For all undefined terms or notions on simplicial complexes we refer the reader to the books [13] or [27].

Given a simplicial complex Δ on $[n]$, the **link**, **star** and **deletion** of σ in Δ are defined, respectively, by

$$\begin{aligned} \text{link}_\Delta(\sigma) &= \{F \in \Delta : \sigma \cap F = \emptyset, \sigma \cup F \in \Delta\}, \\ \text{star}_\Delta(\sigma) &= \{F \in \Delta : \sigma \cup F \in \Delta\} \text{ and} \\ \Delta \setminus \sigma &= \{F \in \Delta : \sigma \not\subseteq F\}. \end{aligned}$$

Moreover, the **Alexander dual** of Δ is defined as $\Delta^\vee = \{F \in \Delta : [n] \setminus F \notin \Delta\}$.

Let $I \subset S$ be a squarefree monomial ideal generated by monomials of degree at least 2. Then there exists a simplicial complex Δ on $[n]$ such that $I = I_\Delta$. The Alexander dual of I is defined $I^\vee = I_{\Delta^\vee}$.

Definition 1.1. [31] Let Δ be a simplicial complex on vertex set $[n]$. Then a face $\sigma \in \Delta$ is called a **shedding face** if it satisfies the following property:

$$\text{no facet of } (\text{star}_\Delta \sigma) \setminus \sigma \text{ is a facet of } \Delta \setminus \sigma.$$

Definition 1.2. [31] A $(d-1)$ -dimensional simplicial complex Δ is recursively defined to be **k -decomposable** if either Δ is a simplex or else has a shedding face σ with $\dim(\sigma) \leq k$ such that both $\text{link}_\Delta \sigma$ and $\Delta \setminus \sigma$ are k -decomposable.

We consider the complexes $\{\}$ and $\{\emptyset\}$ to be k -decomposable for $k \geq -1$. Also k -decomposability implies to k' -decomposability for $k' \geq k$.

A 0-decomposable simplicial complex is called **vertex-decomposable**.

We say that the simplicial complex Δ is (non-pure) **shellable** if its facets can be ordered F_1, F_2, \dots, F_r such that, for all $r \geq 2$, the subcomplex $\langle F_1, \dots, F_{j-1} \rangle \cap \langle F_j \rangle$

is pure of dimension $\dim(F_j) - 1$ [3]. It was shown in [31] or [18] that a $(d - 1)$ -dimensional (not necessarily pure) simplicial complex Δ is shellable if and only if it is $(d - 1)$ -decomposable.

Let I be a monomial ideal of S . We denote by $G(I)$ the set of minimal monomial generators of I . Let $\min(I)$ be the set of minimal (under inclusion) prime ideals of S containing I .

For $\mathbf{a} \in \mathbb{N}^n$, set $\mathbf{x}^{\mathbf{a}} = \prod_{\mathbf{a}(i) > 0} x_i^{\mathbf{a}(i)}$ and define the **support** of \mathbf{a} by $\text{supp}(\mathbf{a}) = \{i : \mathbf{a}(i) > 0\}$. We set $\text{supp}(\mathbf{x}^{\mathbf{a}}) := \text{supp}(\mathbf{a})$. Also, we define $\bar{\mathbf{a}}$ an n -tuple in $\{0, 1\}^n$ with $\bar{\mathbf{a}}(i) = 1$ if $\mathbf{a}(i) \neq 0$ and $\bar{\mathbf{a}}(i) = 0$, otherwise. Set $\nu_i(\mathbf{x}^{\mathbf{a}}) := \mathbf{a}(i)$.

Let $u, v \in S$ be two monomials. We set $[u, v] = 1$ if for all $i \in \text{supp}(u)$, $x_i^{a_i} \nmid v$ and $[u, v] \neq 1$, otherwise.

For the monomial $u \in S$ and the monomial ideal $I \subset S$ set

$$I^u = \langle v \in G(I) : [u, v] \neq 1 \rangle \quad \text{and} \quad I_u = \langle v \in G(I) : [u, v] = 1 \rangle.$$

Definition 1.3. [23] Let I be a monomial ideal with the minimal system of generators $\{u_1, \dots, u_r\}$. The monomial $v = x_1^{a_1} \dots x_n^{a_n}$ is called **shedding** if $I_v \neq 0$ and for each $u_i \in G(I_v)$ and each $l \in \text{supp}(u)$ there exists $u_j \in G(I^v)$ such that $u_j : u_i = x_l$.

Definition 1.4. [23] Let I be a monomial ideal minimally generated with set $\{u_1, \dots, u_r\}$. We say I is a **k -decomposable** ideal if $r = 1$ or else has a shedding monomial v with $|\text{supp}(v)| \leq k + 1$ such that the ideals I^v and I_v are k -decomposable. (Note that since the number of minimal generators of I is finite, the recursion procedure will stop.)

A 0-decomposable monomial ideal is called **variable-decomposable**.

Theorem 1.5. [23, Theorem 2.10.] *Let Δ be a (not necessarily pure) $(d - 1)$ -dimensional simplicial complex on vertex set $[n]$. Then Δ is k -decomposable if and only if I_{Δ^\vee} is k -decomposable, where $k \leq d - 1$.*

Definition 1.6. [20] A monomial ideal I is called **weakly polymatroidal** if for every two monomials $u = x_1^{a_1} \dots x_n^{a_n} >_{\text{lex}} v = x_1^{b_1} \dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$.

Theorem 1.7. [24, Theorem 4.33.] *Every weakly polymatroidal ideal I is variable-decomposable.*

2. k -CLEAN MONOMIAL IDEALS

In this section we extend the concept of cleanness introduced by Dress [7]. Let $I \subset S$ be a monomial ideal. A prime filtration

$$\mathcal{F} : (0) = M_0 \subset M_1 \subset \dots \subset M_{r-1} \subset M_r = S/I$$

of S/I is called **multigraded**, if all M_i are multigraded submodules of S/I , and if there are multigraded isomorphisms $M_i/M_{i-1} \cong S/P_i(-\mathbf{a}_i)$ with some $\mathbf{a}_i \in \mathbb{Z}^n$ and some multigraded prime ideals P_i .

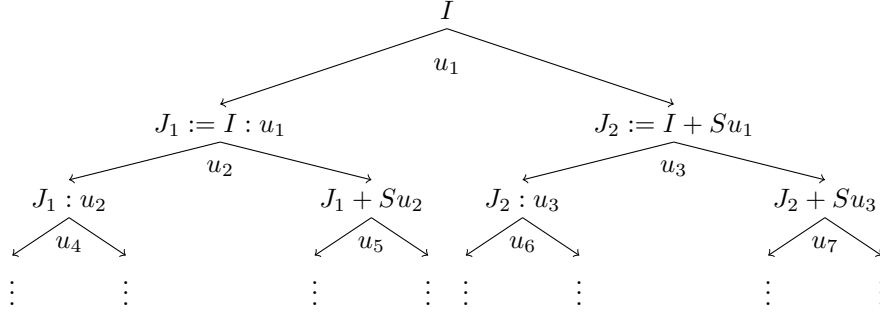
A multigraded prime filtration \mathcal{F} of S/I is called **clean** if $\text{Supp}(\mathcal{F}) \subseteq \min(I)$.

Definition 2.1. Let $I \subset S$ be a monomial ideal. A non unit monomial $u \notin I$ is called a **cleaner** monomial of I if $\min(I + Su) \subseteq \min(I)$.

Definition 2.2. Let $I \subset S$ be a monomial ideal. We say that I is **k -clean** whenever I is a prime ideal or I has no embedded prime ideals and there exists a cleaner monomial $u \notin I$ with $|\text{supp}(u)| \leq k+1$ such that both $I : u$ and $I + Su$ are k -clean.

We recall the concept of ideal tree from [25]:

Let $I \subset S$ be a k -clean monomial ideal. By the definition, there are cleaner monomials u_1, u_2, \dots with $|\text{supp}(u_i)| \leq k+1$ decomposing I . Therefore we obtain the rooted, finite, directed and binary tree \mathcal{T} :



\mathcal{T} is called the **ideal tree** of I and the number of all cleaner monomials appeared in \mathcal{T} is called the **length** of \mathcal{T} . We denote the length of \mathcal{T} by $l(\mathcal{T})$.

We define the **k -cleanness length** of the k -clean monomial ideal I by

$$l(I) = \min\{l(\mathcal{T}) : \mathcal{T} \text{ is an ideal tree of } I\}.$$

Example 2.3. Consider the monomial ideal

$$I = (x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3x_5, x_1x_3x_6, x_1x_4x_5, x_2x_3x_6, \\ x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6)$$

and

$$J = (x_1x_2, x_1x_3, x_1x_4)$$

of the polynomial ring $S = K[x_1, \dots, x_6]$. I and J are, respectively, 1-clean and 0-clean and have ideal trees \mathcal{T}_1 and \mathcal{T}_2 such that the cleaner monomials appeared in \mathcal{T}_1 and \mathcal{T}_2 are, respectively, $x_2x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_2, x_1, x_3x_6, x_3$ and x_1 .

Theorem 2.4. Every k -clean monomial ideal I is clean. Also, every clean monomial ideal is k -clean for some $k \geq 0$.

Proof. Let I be a k -clean monomial ideal. We use induction on the k -cleanness length of I . Let I be not prime and there exists a cleaner monomial $u \notin I$ of multidegree \mathbf{a} with $|\text{supp}(u)| \leq k+1$ such that both $I : u$ and $I + Su$ are k -clean. By induction, $I : u$ and $I + Su$ are clean. Let

$$\mathcal{F}_1 : I + Su = J_0 \subset J_1 \subset \dots \subset J_r = S$$

and

$$\mathcal{F}_2 : 0 = \frac{L_0}{I : u} \subset \frac{L_1}{I : u} \subset \dots \subset \frac{L_s}{I : u} = \frac{S}{I : u}.$$

be clean prime filtrations and let $(L_i/I : u)/(L_{i-1}/I : u) \cong S/Q_i(-\mathbf{a}_i)$ where Q_i are prime ideals. It is known that the multiplication map $\varphi : S/I : u(-\mathbf{a}) \xrightarrow{\cdot u} I + Su/I$ is an isomorphism. Restricting φ to $L_i/I : u$ yields a monomorphism $\varphi_i : L_i/I :$

$u \xrightarrow{\cdot u} I + Su/I$. Set $H_i/I := \varphi_i(L_i/I : u)$. Hence $H_i/I \cong (L_i/I : u)(-\mathbf{a})$. It follows that

$$\frac{H_i}{H_{i-1}} \cong \frac{H_i/I}{H_{i-1}/I} \cong \frac{(L_i/I : u)(-\mathbf{a})}{(L_{i-1}/I : u)(-\mathbf{a})} \cong \frac{S}{Q_i}(-\mathbf{a} - \mathbf{a}_i).$$

Therefore we obtain the following prime filtration induced from \mathcal{F}_2 :

$$\mathcal{F}_3 : I = H_0 \subset H_1 \subset \dots \subset H_s = I + Su.$$

By adding \mathcal{F}_1 to \mathcal{F}_3 we obtain the following prime filtration

$$\mathcal{F} : I = H_0 \subset H_1 \subset \dots \subset H_s = I + Su \subset J_1 \subset \dots \subset J_r = S.$$

Finally, $\text{Supp}(\mathcal{F}) = \text{Supp}(\mathcal{F}_1) \cup \text{Supp}(\mathcal{F}_2) \subset \min(I + Su) \cup \min(I : u) \subseteq \min(I)$ and therefore I is clean.

To prove the second assertion, suppose that I is a clean monomial ideal. If I is prime then we are done. Suppose that I is not prime and let

$$\mathcal{F} : (0) = M_0 \subset M_1 \subset \dots \subset M_{r-1} \subset M_r = S/I$$

be a clean prime filtration of S/I with $M_i/M_{i-1} \cong S/P_i(-\mathbf{a}_i)$. We use induction on the length of the prime filtration \mathcal{F} . Since that $\text{Ass}(S/I) \subseteq \text{Supp}(\mathcal{F}) \subseteq \min(I)$, we have $\text{Ass}(S/I) = \min(I)$. Hence I has no embedded prime ideal. It follows from Proposition 10.1. of [15] that there is a chain of monomial ideals $I = I_0 \subset I_1 \subset \dots \subset I_r = S$ and monomials u_i of multidegree \mathbf{a}_i such that $I_i = I_{i-1} + Su_i$ and $I_{i-1} : u_i = P_i$. Since that $I + Su_1$ has a clean filtration, it is k -clean, by induction hypothesis, where $|\text{supp}(u_1)| \leq k + 1$. On the other hand, $I + Su_1/I \cong S/P_1$. Therefore $\min(I + Su_1) = \{P_1\} \subset \min(I)$. This means that I is k -clean. \square

3. SOME PROPERTIES OF k -CLEAN MONOMIAL IDEALS

Theorem 3.1. *Let $I \subset S$ be k -clean. Then for all monomial $u \in S$, $I : u$ is k -clean.*

Proof. We use induction on the k -cleanness length of I . If I is prime then $I : u$ is prime, too and we have nothing to prove. Assume that I is not prime. Suppose v is a cleaner monomial of I with $|\text{supp}(v)| \leq k + 1$ and $I : v$ and $I + (v)$ are k -clean. We consider two cases:

Case 1. Let $v|u$. Then $I : u = (I : v) : u/v$ and by induction hypothesis $I : u$ is k -clean.

Case 2. Let $v \nmid u$. We show that $v/\text{gcd}(u, v)$ is a cleaner monomial of $I : u$. We have

$$(I : u) + \left(\frac{v}{\text{gcd}(u, v)}\right) = (I + (v)) : u \quad \text{and} \quad (I : u) : \frac{v}{\text{gcd}(u, v)} = (I : v) : \frac{u}{\text{gcd}(u, v)}.$$

By induction, $(I : u) + \left(\frac{v}{\text{gcd}(u, v)}\right)$ and $(I : u) : \frac{v}{\text{gcd}(u, v)}$ are k -clean. Since $\min(I + (v)) \subset \min(I)$, by some elementary computations, we obtain that $\min((I + (v)) : u) \subset \min(I : u)$. Therefore $v/\text{gcd}(u, v)$ is a cleaner monomial of $I : u$. \square

Theorem 3.2. *The radical of each k -clean monomial ideal is k -clean.*

Proof. Let $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r})$ be a k -clean monomial ideal with cleaner monomial $u = \mathbf{x}^{\mathbf{b}}$ with $|\text{supp}(u)| \leq k + 1$. We use induction on the k -cleanness length of I . Denote the radical of I by \sqrt{I} . By induction hypothesis, $\sqrt{I + Su}$ and $\sqrt{I} : u$ are k -clean. Let $v = \mathbf{x}^{\text{supp}(u)}$ and let w be the product of variables x_i with $i \in \text{supp}(u)$ and $\mathbf{a}_j(i) > \mathbf{b}(i) > 0$ for some $1 \leq j \leq r$. $\sqrt{I} + Sv$ is k -clean, because $\sqrt{I} + Sv =$

$\sqrt{I+Su}$. Also, $\sqrt{I} : v = (\sqrt{I} : u) : w$ and so $\sqrt{I} : v$ is k -clean, by Theorem 3.1. On the other hand, $\min(\sqrt{I}+Sv) \subset \min(\sqrt{I}+Su) = \min(I+Su) \subset \min(I) = \min(\sqrt{I})$ and so v is a cleaner monomial of \sqrt{I} . \square

Let $u = x_{i_1}^{a_1} \dots x_{i_t}^{a_t} \in S$. The **polarization** of u is defined by

$$u^p = x_{i_1 1} \dots x_{i_1 a_1} \dots x_{i_t 1} \dots x_{i_t a_t}.$$

If $I \subset S$ is a monomial ideal. The polarization of I is a monomial ideal of $S^p = K[x_{ij} : x_{ij}|u^p \text{ for some } u \in G(I)]$ given by $I^p = (u^p : u \in G(I))$.

Define the K -algebra homomorphism $\pi : S^p \rightarrow S$ by $\pi(x_{ij}) = x_i$.

Theorem 3.3. *Let I be a monomial ideal with no embedded prime ideal. If I^p is k -clean then I is k -clean, too.*

Proof. We use induction on the k -cleanness length of I^p . If I is a prime ideal then we have nothing to prove. Suppose that I is not prime. Let u be a cleaner monomial of I^p with $|\text{supp}(u)| \leq k+1$ and let $I^p : u$ and $I^p + (u)$ be k -clean. We claim that $\pi(u)$ is a cleaner monomial of I . Note that

$$I : \pi(u) = \pi(I^p : u) \text{ and } I + (\pi(u)) = \pi(I^p + (u)).$$

By induction hypothesis, $I : \pi(u)$ and $I + (\pi(u))$ are k -clean. Since $|\text{supp}(\pi(u))| \leq |\text{supp}(u)| \leq k+1$, it remains to show that $\pi(u)$ is a cleaner monomial of I . Let $P \in \min(I + (\pi(u)))$. Hence there exists $Q \in \min(I^p + (u))$ such that $P = \pi(Q)$. Since $Q \in \min(I^p)$, it follow that $P \in \min(I)$, as desired. \square

Lemma 3.4. *Let $I \subset S$ be a k -clean monomial ideal with cleaner monomial u . Then u^p is a cleaner monomial of I^p .*

Proof. Let $Q \in \min(I^p + (u^p))$. Then $Q \in \text{Ass}(S^p/I^p + (u^p))$. By Corollary 2.6 of [9], $\pi(Q) \in \text{Ass}(S/I + (u)) = \min(I + (u)) \subset \min(I)$. Again, by Proposition 2.3 of [9], $Q \in \min(I^p)$, as desired. \square

The following theorem describes projective dimension and Castelnuovo-Mumford regularity of k -clean monomial ideals

Theorem 3.5. *Let $I \subset S$ be a k -clean monomial ideal with the cleaner monomial u . Then*

- (i) $\text{pd}(S/I) = \max\{\text{pd}(S/I + (u)), \text{pd}(S/I : u)\};$
- (ii) $\text{reg}(S/I) = \max\{\text{reg}(S/I + (u)), \text{reg}(S/I : u) + \deg(u)\}.$

Proof. (i) Without loss of generality we may assume that $I \subset \mathfrak{m}^2$. By Corollary 1.6.3. of [13], $\text{pd}(S/I) = \text{pd}(S^p/I^p)$ and $\text{reg}(S/I) = \text{reg}(S^p/I^p)$. Let Δ be a simplicial complex with $I_\Delta = I^p$. By Lemma 3.4, u^p is a cleaner monomial of I^p . Let $u^p = \mathbf{x}^\sigma$ for some $\sigma \in \Delta$. Therefore Δ is a k -decomposable simplicial complex with shedding monomial σ , by Theorem 4.1. Now it follows from Theorem 2.8 of [21] that

$$\begin{aligned} \text{pd}(S/I) &= \text{pd}(S^p/I_\Delta) = \max\{\text{pd}(S^p/I_{\Delta \setminus \sigma}), \text{pd}(S^p/J_{\text{link}_\Delta \sigma})\} \\ &= \max\{\text{pd}(S^p/(I + (u))^p), \text{pd}(S^p/(I : u)^p)\} \\ &= \max\{\text{pd}(S/I + (u)), \text{pd}(S/I : u)\} \end{aligned}$$

where $J_{\text{link}_\Delta \sigma}$ is the Stanley-Reisner ideal of $\text{link}_\Delta \sigma$ considered as a complex on $V(\Delta) \setminus \sigma$.

(ii) follows by a similar argument from Theorem 2.8 of [21] and Theorem 4.1. \square

Remark 3.6. The concept of sequentially Cohen-Macaulayness was introduced in [27] for finitely generated (graded) modules. We specially recall this concept for the quotient rings. Let $I \subset S$ be a monomial ideal. We say that I is sequentially Cohen-Macaulay if there exists a finite filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_r = S/I$$

of submodules of S/I with these properties that M_i/M_{i-1} is Cohen-Macaulay and

$$\dim(M_1/M_0) \leq \dim(M_2/M_1) \leq \dots \leq \dim(M_r/M_{r-1}).$$

It was proven in [15] that cleanness implies sequentially Cohen-Macaulayness. Therefore the class of k -clean monomial ideals is contained in the class of sequentially Cohen-Macaulay monomial ideals. In particular, since that every unmixed sequentially Cohen-Macaulay monomial ideal is Cohen-Macaulay, we conclude that the unmixed k -clean monomial ideals are Cohen-Macaulay.

4. A VIEW TOWARD k -DECOMPOSABLE SIMPLICIAL COMPLEXES

In this section, we prove the main result of this paper. In fact, we show that a squarefree k -clean monomial ideal is Stanley-Reisner ideal of a k -decomposable simplicial complex, and vice versa.

Theorem 4.1. *Let Δ be a $(d-1)$ -dimensional simplicial complex. Then $\sigma \in \Delta$ is a shedding face of Δ if and only if \mathbf{x}^σ is a cleaner monomial of I_Δ .*

In particular, Δ is k -decomposable if and only if I_Δ is k -clean, where $0 \leq k \leq d-1$.

Proof. We first show that σ is a shedding face of Δ if and only if $\min(I_\Delta + (\mathbf{x}^\sigma)) \subseteq \min(I_\Delta)$. Since that Stanley-Reisner rings are reduced, it follows that

$$\min(I_\Delta) = \{P_{F^c} : F \in \mathcal{F}(\Delta)\}$$

and

$$\min(I_\Delta + (\mathbf{x}^\sigma)) = \{P_{F^c} : F \in \mathcal{F}(\Delta \setminus \sigma)\}.$$

Let σ be the shedding face of Δ . To show that \mathbf{x}^σ is a cleaner monomial of I_Δ , it suffices to prove $\mathcal{F}(\Delta \setminus \sigma) \subseteq \mathcal{F}(\Delta)$. Suppose, on the contrary, that $F \in \mathcal{F}(\Delta \setminus \sigma)$ and $F \subsetneq G$ with $G \in \mathcal{F}(\Delta)$. This implies that $\sigma \subset G$ and so $G \in \text{star}_\Delta \sigma$. On the other hand, since F is a facet of $\Delta \setminus \sigma$, it follows that there is $t \in \sigma$ such that $\sigma \setminus \{t\} \subset F$. We claim that $G = F \dot{\cup} \{t\}$. The inclusion “ \supseteq ” is clear. For the converse inclusion, if $s \in G \setminus (F \cup \{t\})$ for some s , then $\sigma \not\subseteq F \cup \{s\}$ and so $F \cup \{s\} \in \mathcal{F}(\Delta \setminus \sigma)$, a contradiction. Therefore $G = F \dot{\cup} \{t\}$ and it follows that $F \in \mathcal{F}((\text{star}_\Delta \sigma) \setminus \sigma)$. But this contradicts the assumption that σ is a shedding face of Δ . Hence \mathbf{x}^σ is a cleaner monomial.

Let Δ be k -decomposable with the shedding face $\sigma \in \Delta$. By the first part, \mathbf{x}^σ is a cleaner monomial of I_Δ . To showing that I_Δ is k -clean, we use induction on the number of the facets of Δ . If Δ is a simplex then the assertion is trivial. So assume that $|\mathcal{F}(\Delta)| > 1$. It is easy to check that $J_{\text{link}_\Delta \sigma} = I_\Delta : \mathbf{x}^\sigma$ and $I_{\Delta \setminus \sigma} = I_\Delta + (\mathbf{x}^\sigma)$. By induction hypothesis, $\text{link}_\Delta \sigma$ and $\Delta \setminus \sigma$ are k -decomposable if and only if $I_\Delta : \mathbf{x}^\sigma$ and $I_\Delta + (\mathbf{x}^\sigma)$ are k -clean. Therefore I_Δ is k -clean.

The reverse directions of both parts follow easily in similar arguments. \square

Remark 4.2. Note that a k -clean monomial ideal need not be k' -clean for $k' < k$. Consider the monomial ideal $I \subset K[x_1, \dots, x_6]$ with the minimal generator set

$$G(I) = \{x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3x_5, x_1x_3x_6, x_1x_4x_5, \\ x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6\}.$$

I is the Stanley-Reisner ideal of the simplicial complex

$$\Delta = \langle 124, 125, 126, 135, 136, 145, 236, 245, 256, 345, 346 \rangle$$

on [6]. It was shown in [25] that Δ is shellable but not vertex-decomposable. It follows from Theorem 4.1 that I is clean but not 0-clean. To see more examples of clean ideals which are not 0-clean we refer the reader to [11, 22].

Remark 4.3. Let I be a clean monomial ideal and $\dim(S/I) = d$. By Theorem 2.4, I is k -clean for some $k \geq 0$ with cleaner monomial u . It follows from Theorem 3.2 that \sqrt{I} is k -clean with cleaner monomial $v = \mathbf{x}^{\text{supp}(u)}$. Let $I_\Delta = \sqrt{I}$ for some simplicial complex Δ on $[n]$. By Theorem 4.1, we have $|\text{supp}(u)| = |\text{supp}(v)| \leq \dim(\Delta) + 1 = d$. Therefore I is $(d-1)$ -clean.

On the other hand, every k -clean monomial ideal is also $(k+1)$ -clean. This means that the k -cleanness is a hierarchical structure. Therefore we have the following implications:

$$0\text{-clean} \Rightarrow 1\text{-clean} \Rightarrow \dots \Rightarrow (d-1)\text{-clean} \Leftrightarrow \text{clean}.$$

In Remark 4.2 we implied that above implications are strict.

Corollary 4.4. *Let $I \subset S$ be a squarefree monomial ideal generated by monomials of degree at least 2. Then I is k -clean if and only if I^\vee is k -decomposable.*

Proof. Let Δ be a simplicial complex on $[n]$ such that $I = I_\Delta$. The assertion follows from Theorems 4.1 and 1.5. \square

5. SOME CLASSES OF k -CLEAN IDEALS

In this section, we introduce some classes of k -clean monomial ideals.

5.1. Irreducible monomial ideals.

Theorem 5.1. *Every irreducible monomial ideal is 0-clean.*

Proof. Let I be an irreducible monomial ideal. We want to show that I is 0-clean. By Theorem 1.3.1. of [13], I is generated by pure powers of the variables. Without loss of generality we may assume that $I = (x_1^{a_1}, \dots, x_m^{a_m})$ with $a_i \neq 0$ for all i . We use induction on $\sum_{i=1}^m a_i$. If $\sum_{i=1}^m a_i = m$, then I is prime and we are done. Suppose that $\sum_{i=1}^m a_i > m$. So we can assume that $a_1 > 1$. We have

$$I : x_1 = (x_1^{a_1-1}, x_2^{a_2}, \dots, x_m^{a_m}) \text{ and } I + (x_1) = (x_1, x_2^{a_2}, \dots, x_m^{a_m}).$$

By induction hypothesis, $I : x_1$ and $I + (x_1)$ are 0-clean. Clearly, x_1 is a cleaner monomial and so the proof is completed. \square

5.2. Monomial complete intersection ideals.

Theorem 5.2. *Let $I \subset S$ be a monomial complete intersection ideal. Then S/I is 0-clean.*

Proof. Let $G(I) = \{M_1, \dots, M_r\}$. By the assumption M_1, \dots, M_r is a regular sequence. Hence $\gcd(M_i, M_j) = 1$ for all $i \neq j$. If I is a primary ideal then we are done, by Theorem 5.1. Suppose that I is not primary. We use induction on n the number of variables. Let $|\text{supp}(M_1)| > 1$ and let $\nu_1(M_1) = a$. Then

$$I : x_1^a = (M_1/x_1^a, M_2, \dots, M_r) \text{ and } I + (x_1^a) = (x_1^a, M_2, \dots, M_r).$$

Since that $(M_1/x_1^a, M_2, \dots, M_r)$ and (x_1^a, M_2, \dots, M_r) are complete intersection monomial ideals with the number of variables less than n , we deduce that $I : x_1^a$ and $I + (x_1^a)$ are 0-clean, by induction hypothesis. Set $J := (M_2, \dots, M_r)$. Since that

$$\min(I + (x_1^a)) = \{P + (x_1) : P \in \min(J)\}$$

and

$$\min(I) = \{P + (x_i) : P \in \min(J) \text{ and } x_i | M_1\}.$$

we conclude that $\min(I + (x_1^a)) \subset \min(I)$ and so x_1^a is a cleaner monomial. \square

5.3. Cohen-Macaulay monomial ideals of codimension 2. Proposition 2.3 from [14] says that if $I \subset S$ is a squarefree monomial ideal with 2-linear resolution, then after suitable renumbering of the variables, one has the following property:

if $x_i x_j \in I$ with $i \neq j$, $k > i$ and $k > j$, then either $x_i x_k$ or $x_j x_k$ belongs to I .

Let I has a 2-linear resolution and the monomials in $G(I)$ be ordered by the lexicographical order induced by $x_n > x_{n-1} > \dots > x_1$. Let $u = x_s x_t > v = x_i x_j$ be squarefree monomials in $G(I)$ with $s < t$ and $i < j$. We have $t \geq j$. If $t = j$, then $x_s(v/x_i) = u \in G(I)$. If $t > j$ then by the above property either $x_i x_t \in G(I)$ or $x_j x_t \in G(I)$. This immediately implies the following lemma.

Lemma 5.3. *If I is a squarefree monomial ideal generated in degree 2 which has a linear resolution, then after suitable renumbering of the variables, I is weakly polymatroidal.*

Theorem 5.4. *Let $I \subset S$ be a monomial ideal which is Cohen-Macaulay and of codimension 2. Then S/I is 0-clean.*

Proof. Since I has no embedded prime ideals, if we show that I^p is 0-clean then it follows from Theorem 3.3 that I is 0-clean. Let Δ be a simplicial complex with $I_\Delta = I^p$. Since I is Cohen-Macaulay, by Corollary 1.6.3. of [13], I_Δ is Cohen-Macaulay, too. In particular, I_Δ^\vee has linear resolution, by the Eagon-Reiner theorem [8]. It follows from Lemma 5.3 and Theorems 1.7 and 4.1 that $I_\Delta = I^p$ is 0-clean, as desired. \square

5.4. Symbolic powers of Stanley-Reisner ideals of matroid complexes. Let Δ be a simplicial complex and let $I_\Delta^{(m)}$ denote the m th symbolic power of I_Δ . Minh and Trung [19] and Varbaro [30] independently proved that Δ is a matroid if and only if $I_\Delta^{(m)}$ is Cohen-Macaulay for all $m \in \mathbb{N}$. Later, in [28], Terai and Trung showed that Δ is a matroid if and only if $I_\Delta^{(m)}$ is Cohen-Macaulay for some integer $m \geq 3$. Recently, Bandari and Soleyman Jahan [1] proved that if Δ is a matroid, then $I_\Delta^{(m)}$ is clean for all $m \in \mathbb{N}$. In this section, we improve this result by showing that if Δ is a matroid, then $I_\Delta^{(m)}$ is 0-clean for all $m \in \mathbb{N}$.

Theorem 5.5. *Let Δ be a matroid complex with $I = I_\Delta$. Then for all $m \geq 1$, $I^{(m)}$ is 0-clean.*

Proof. Let $\Delta = \langle F_1, \dots, F_t \rangle$. Then $I = I_\Delta = \bigcap_{i=1}^t P_{F_i^c}$ and $(I_\Delta)^{(m)} = \bigcap_{i=1}^t (P_{F_i^c})^{(m)}$. Since Δ is a matroid and I is Cohen-Macaulay, it follows that $I^{(m)}$ has no embedded prime ideal. Therefore if we show that $(I^{(m)})^p$ is 0-clean then the proof is completed, by Theorem 3.3.

In [1] the authors introduced an ordering on the variables of S^p and showed that $((I^{(m)})^p)^\vee$ has linear quotients with respect to this ordering. We improve this result by considering the same ordering to show that $((I^{(m)})^p)^\vee$ is weakly polymatroidal. Then by Theorem 1.7 and Corollary 4.4, $(I^{(m)})^p$ is 0-clean. We use some notations of the proof of [1, Theorem 2.1.]. It is known that Δ^c is a matroid. Let $\dim(\Delta^c) = r - 1$. We set $J = ((I^{(m)})^p)^\vee$. Then

$$G(J) = \{x_{i_1, j_1} x_{i_2, j_2} \dots x_{i_r, j_r} : \{i_1, \dots, i_r\} \text{ is a facet of } \Delta^c\}$$

where $1 \leq j_l \leq m$ and $\sum_{l=1}^r j_l \leq m + r - 1$.

Consider the order $<$ on the variables of S^α by setting $x_{i, j} > x_{i', j'}$ if either $j < j'$, or $j = j'$ and $i < i'$. Let $u, v \in G(J)$ with $u = x_{i_r, j_r} \dots x_{i_2, j_2} x_{i_1, j_1} > v = x_{i'_r, j'_r} \dots x_{i'_2, j'_2} x_{i'_1, j'_1}$ such that $x_{i_l, j_l} = x_{i'_l, j'_l}$ for all $l > t$ and $x_{i_t, j_t} > x_{i'_t, j'_t}$. We have two cases:

Case 1. $x_{i_t} | x_{i'_r} \dots x_{i'_{t+1}} x_{i'_t}$. Let $i'_l = i_t$. It is clear that $j_t < j'_l$. In particular, $x_{i_t, j_t} (v / x_{i'_l, j'_l}) \in G(J)$.

Case 2. $x_{i_t} \nmid x_{i'_r} \dots x_{i'_{t+1}} x_{i'_t}$. Since I_{Δ^\vee} is matroidal, it follows from [12, Lemma 3.1.] that there exists $i'_l \notin \{i_1, \dots, i_r\}$ such that $x_{i_t} (x_{i'_r} \dots x_{i'_1} / x_{i'_l}) \in I_{\Delta^\vee}$. Therefore

$$x_{i'_r, j'_r} \dots x_{i'_{l-1}, j'_{l-1}} x_{i'_{l+1}, j'_{l+1}} \dots x_{i_t, j_t} x_{i'_{t-1}, j'_{t-1}} \dots x_{i'_1, j'_1} \in G(J).$$

Therefore J is weakly polymatroidal, as desired. \square

It follows from Theorem 5.5 that we can add the condition “0-cleanness of $I_{\Delta}^{(m)}$ for all $m > 0$ ” to [1, Corollary 2.3.]:

Corollary 5.6. *Let Δ be a pure simplicial complex and $I = I_{\Delta} \subset S$. Then the following conditions are equivalent:*

- (i) Δ is a matroid;
- (ii) $S/I^{(m)}$ is 0-clean for all integer $m > 0$;
- (iii) $S/I^{(m)}$ is clean for some integer $m > 0$;
- (iv) $S/I^{(m)}$ is clean for some integer $m \geq 3$;
- (v) $S/I^{(m)}$ is Cohen-Macaulay for some integer $m \geq 3$;
- (vi) $S/I^{(m)}$ is Cohen-Macaulay for all integer $m > 0$.

Cowsik and Nori in [6] proved that for any homogeneous radical ideal I in the polynomial ring S , all the powers of I are Cohen-Macaulay if and only if I is a complete intersection. We call the simplicial complex Δ **complete intersection** if I_{Δ} is a complete intersection ideal. Therefore the simplicial complex Δ is a complete intersection if and only if I_{Δ}^m is Cohen-Macaulay for any $m \in \mathbb{N}$ ([29, Theorem 3]). We improve this result in the following. By the fact that if I_{Δ}^m is Cohen-Macaulay then I_{Δ}^m is equal to the m th symbolic power $I_{\Delta}^{(m)}$ of I_{Δ} we have

Corollary 5.7. *Let Δ be a pure simplicial complex and $I = I_{\Delta} \subset S$. Then the following conditions are equivalent:*

- (i) Δ is a complete intersection;
- (ii) S/I^m is 0-clean for all integer $m > 0$;
- (iii) S/I^m is clean for some integer $m > 0$;
- (iv) S/I^m is clean for some integer $m \geq 3$;
- (v) S/I^m is Cohen-Macaulay for some integer $m \geq 3$;
- (vi) S/I^m is Cohen-Macaulay for all integer $m > 0$.

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